March 2000

UTAS-PHYS-00-02

arXiv:quant-ph/0002090 v2 13 Mar 2000

# Integrity bases for local invariants of composite quantum systems: Corrigendum to J Phys A33 (2000) 1895-1914

R I A Davis*, R Delbourgo, P D Jarvis

*School of Mathematics and Physics, University of Tasmania*
*GPO Box 252-21, Hobart Tas 7001, Australia*

## Abstract

Unitary group branchings appropriate to the calculation of local invariants of density matrices of composite quantum systems are formulated using the method of $S$-function plethysms. From this, the generating function for the number of invariants at each degree in the density matrix can be computed. For the case of two two-level systems, the generating function is $F(q) = 1 + q + 4q^2 + 6q^3 + 16q^4 + 23q^5 + 52q^6 + 77q^7 + 150q^8 + 224q^9 + 396q^{10} + 583q^{11} + O(q^{12})$. Factorisation of such series leads in principle to the identification of an integrity basis of algebraically independent invariants. This note replaces Appendix B of our paper[1] J Phys **A33** (2000) 1895-1914 (`quant-ph/0001076`) which is incorrect.

* Dept of Physics, University of Queensland, St Lucia Brisbane 4072

The measurement problem of detecting nonlocal differences between composite quantum systems (for example, degrees of entanglement[1]) is of great importance for applications to quantum computation and communication. At root the question boils down to the identification of invariants with respect to unitary transformations which can be effected by local operations on each subsystem separately. For the case of two subsystems of dimensions $N_1$ and $N_2$, the $N_1N_2\times N_1N_2$ density matrix $\rho$ can be regarded in partition labelling[2] as an element of the defining representation $\{1\}$ of $U({N_1}^2{N_2}^2)$ branching to the reducible $\{\bar{1}\}\{1\}\times\{\bar{1}\}\{1\}$ representation of $U(N_1)\times U(N_2)$ via $(\{\bar{1}\}\times\{1\})\times(\{\bar{1}\}\times\{1\})$ of $(U(N_1)\times U(N_1))\times(U(N_2)\times U(N_2))$ within $\{1\}\times\{1\}$ of $U({N_1}^2)\times U({N_2}^2)$. Polynomial invariants of degree $n\geq 0$ are thus $U(N_1)\times U(N_2)$ singlets of the totally symmetric Kronecker power $\{1\}\otimes\{n\}\equiv\{n\}$ of $U({N_1}^2{N_2}^2)$. According to the standard rules for plethysms[2, 3] the branchings

$$U({N_1}^2{N_2}^2)\supset U({N_1}^2)\times U({N_2}^2)\supset U(N_1)\times U(N_2) \tag{1}$$

for this plethysm are

$$\begin{aligned}\{1\}\otimes\{n\}=(\{1\}\times\{1\})\otimes\{n\} &= \sum_{\sigma\,\vdash\,n}\{\sigma\}\times\{\sigma\circ n\}\equiv\sum_{\sigma\,\vdash\,n}\{\sigma\}\times\{\sigma\} \\ &= \sum_{\substack{\kappa\,\vdash_{N_1}\,n,\\ \lambda\,\vdash_{N_2}\,n}}\{\bar{\kappa}\}\{\kappa\circ\sigma\}\times\{\bar{\lambda}\}\{\lambda\circ\sigma\}. \end{aligned} \tag{2}$$

Here $\kappa$ and $\lambda$ must be $N_1$, $N_2$ part partitions of $n$ respectively in order that the corresponding representations of $U(N_1)$ and $U(N_2)$ be nonvanishing. However, since the product of two representations in a unitary group will contain a singlet if and only if they are contragredient, the only singlets occurring are those for which both $\kappa\circ\sigma\ni\kappa$ and $\lambda\circ\sigma\ni\lambda$, or reciprocally $\sigma\in\lambda\circ\lambda$ and $\sigma\in\kappa\circ\kappa$. The number of singlets $F_n$ at degree $n$ is thus

$$F_n=\sum_{\substack{\kappa\,\vdash_{N_1}\,n,\\ \lambda\,\vdash_{N_2}\,n}} n_{\kappa\lambda} \tag{3}$$

where $n_{\kappa\lambda}$ counts the number of $\sigma$ satisfying this condition[†]. The computation thus reduces to the evaluation of inner products $\circ$ of $N_1$ and $N_2$ part partitions of $n$ (Kronecker products in the symmetric group $S_n$) and leads to the generating function

$$F(q)=\sum_{n=0}^{\infty}F_nq^n. \tag{4}$$

$F(q)$ is difficult to compute in closed form, but for specific cases can be evaluated to any desired degree[6]. For example at degree 8 for the $2\times 2$ case, we find

$$\{6,2\}\circ\{6,2\} \;=\; \{8\}+\{71\}+2\{62\}+\{61^2\}+\{53\}+2\{521\}+\{51^3\}+\{4^2\}+\{431\}$$

[†] Also $\sigma$ should have at most $\min({N_1}^2,{N_2}^2)$ parts. In Appendix B of our paper[1], $n_{\kappa\lambda}$ was erroneously taken as 1

$$
\begin{aligned}
& + \{42^2\}; \\
\{5,3\} \circ \{5,3\} \;=\; & \{8\} + \{71\} + 2\{62\} + \{61^2\} + \{53\} + 2\{521\} + \{51^3\} + \{4^2\} + 2\{431\} \\
& + 2\{42^2\} + \{421^2\} + \{3^2 2\} + \{3^2 1^2\} + \{32^2 1\}; \qquad (5)
\end{aligned}
$$

leading to a contribution (including multiplicity) of $n_{\{6,2\},\{5,3\}} = n_{\{5,3\},\{6,2\}} = 18$ to $F_8$. In this way we calculate

$$
F(q) = 1+q+4q^2+6q^3+16q^4+23q^5+52q^6+77q^7+150q^8+224q^9+396q^{10}+583q^{11}+O(q^{12}). \qquad (6)
$$

This generating function should be compared with the Molien series[4] defined via group integration,

$$
P(z) = \int_{g\in G} \frac{d\mu_G(g)}{\det(\mathbb{1} - zg)}. \qquad (7)
$$

The equivalence between the two series can be readily established in the $S$-function formalism. Write $g$ as the element of $U(N_1^2 N_2^2)$ corresponding to the adjoint action of $U(N_1)\times U(N_2)$ on $\rho$. Characters of group representations are generated by taking traces $\langle g^n \rangle$ of powers of $g$, and hence are polynomials of the class parameters ($x_1 = \exp i\phi_1, x_2 = \exp i\phi_2, \ldots$). From the integrand of (7) we have directly

$$
\begin{aligned}
[\det(\mathbb{1} - zg)]^{-1} &= \prod_i \frac{1}{(1 - zx_i)} \\
&= \sum_{n=0}^{\infty} z^n S_{\{n\}}(x) \qquad (8)
\end{aligned}
$$

where a standard form of the Cauchy product identity has been used[7] (the complete Schur functions $S_{\{n\}}(x)$ correspond to one part partitions; for one argument $S_{\{n\}}(z) = z^n$). The integrand at degree $n$ thus is indeed the reducible $U(N_1)\times U(N_2)$ character corresponding to the $n$'th symmetrised power $\{n\}$ of the fundamental representation of $U(N_1^2 N_2^2)$, and the invariant integration over $U(N_1)\times U(N_2)$ serves to project the identity representation‡.

Makhlin[5] has recently examined the 2×2 case and proposed a concrete set of 18 local invariants for mixed state density matrices. Definitive confirmation of the completeness of such a set is in principle provided by a factorisation of $F(q)$ which establishes an integrity basis presentation of the algebra of invariants in terms of a number of free generators together with additional relations. Unfortunately (6) is not computed to sufficiently many terms to deduce an unequivocal factorisation, but for example the form

$$
\begin{aligned}
G(x) \;\equiv\; & \frac{(1+x^4)(1+x^5)(1+x^6)^4(1+x^7)^2(1+x^8)^2(1+x^9)^2}{(1-x)(1-x^2)^3(1-x^3)^2(1-x^4)^3} \\
=\; & 1 + x + 4x^2 + 6x^3 + 16x^4 + 23x^5 + 52x^6 + \\
& + 77x^7 + 150x^8 + 224x^9 + 398x^{10} + 589x^{11} + 982x^{12} + O(x^{13})
\end{aligned}
$$

may be noted. This has a denominator set signalling 9 free generators, including 1 at degree 1 (the trace of $\rho$) and 3 at degree 2 (the traces of the squares of $\rho$ and of the

‡ $F(q)$ in (6) agrees with $P(z)$ quoted by [4] up to degree 11

reduced density matrices). The total count of 9 is to be expected from the dimensionality of the coset manifold $SU(4)/S(U(2)\times U(2))$, namely $8 = 15 - (8-1)$ plus the overall singlet trace. To finite degree it is not possible uniquely to identify the denominator factors, and the set of free generators given by $G(x)$ differs from that implied by [5]. Of course, the saturation of terms in $G(x)$ in (9) compared with $F(q)$ in (6) beyond degree 9 also requires that some of the numerator factors (corresponding to invariant quantities whose squares or products are relations in the algebra) should be combined together more economically. Nonetheless, the total count in $G(x)$ of 21 invariants (9 denominator plus 12 numerator quantities), and their degrees, is in agreement with [4].

**Acknowledgement**

The authors thank Prof Brian Wybourne for independently repeating our calculation and correcting an error. Prof Ron King (private communication) has confirmed that the degree 12 coefficient in $F(q)$ is 964 in agreement with [4]. We further thank Dr Markus Grassl for correspondence about our results.

# References

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[2] Partition labelling for group representations is developed in G R E Black, R C King and B G Wybourne, J Phys **A16** (1983) 1555 where extensive rules for representation products and branchings are presented.

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[5] Y Makhlin, *Nonlocal properties of 2-qubit gates and mixed states and optimization of quantum computations* (`quant-ph/0002045`)

[6] For example with the package `SCHUR`, ©Schur software associates 1990.

[7] I G MacDonald, *Symmetric functions and Hall polynomials* (Oxford: Clarendon Press) (1979)